Mathematics 222B Lecture 13 Notes

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1 Maximum Principles for Solutions to Elliptic PDEs

1.1 The weak maximum principle

Today, we will cover maximum principles. This material corresponds to section 6.5 in Evans' textbook. This is a theory for solutions to elliptic PDEs in terms of their pointwise values (inherently *scalar*). Here, it is very important that $u : U \to \mathbb{R}$ is real-valued.

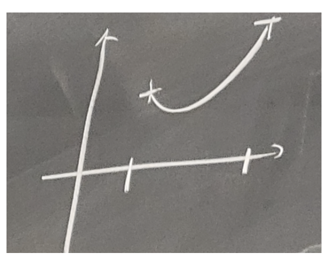
For today's lecture, it is more convenient to consider operators in non-divergence form:

$$Pu = -a^{j,k} \partial_j \partial_k u_k + b^j \partial_j u + cu.$$

We assume the ellipticity condition, that $a \succ \lambda I$ for some $\lambda > 0$, and we assume that $a, b, c \in L^\infty$. (Often, we will start with $c = 0$.)

The theory of maximum principles should be thought of as a generalization of the theory of convex functions on $\mathbb{R}$. In the case of convex functions on $\mathbb{R}$, we have the following theorem.

Theorem 1.1. *Suppose $u : I \to \mathbb{R}$ is convex. Then $\max_I u = \max_{\partial I}$, i.e. the maximum is attained on the boundary.*



One way to generalize 1 dimensional convex functions is to look at convex functions in d dimensions. This is very useful, but it may be too restrictive. Instead, we should think of subsolutions to elliptic PDEs.

Definition 1.1. We say that $u \in C^2(U)$ is a **(classical) subsolution** if $Pu \le 0$.

Remark 1.1. When $d = 1$ and $P = -a\partial_x^2$ with $a > 0$, $Pu \leq 0$ if and only if u is convex.

Theorem 1.2 (Weak maximum principle)**.** *Let U be a connected, bounded, open subset of $\mathbb{R}^d$. Let $u \in C^2(U) \cap C(\overline{U})$ with $Pu \leq 0$. Assume for now that $c = 0$. Then*

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Proof. Step 1: Consider strict subsolutions $Pu < 0$. We will show that no interior maximum is possible. Suppose, for contradiction, that $x_0 \in U$ is a (local) maximum. Then $Du(x_0) = 0$, and the second derivative test tells us that $D^2u(x_0) \leq 0$. We have

$$\begin{aligned} 0 &> Pu(x_0) \\ &= -a^{j,k}\partial_j\partial_k u|_{x=x_0} + b^j \underbrace{\partial_j u|_{x=x_0}}_{=Du=0} + \underbrace{c}_{=0} u \end{aligned}$$

We will interpret the first term as a trace. Call $h = D^2u$. Since a is positive definite, we can find an orthogonal matrix O such that $OaO^{-1} = D$, where D is diagonal with positive entries e_j. This makes $a^{j,k}\partial_j\partial_k = O_{j,j'}e_{j'}\delta_{j',k'}O_{k,k'}$. Then $a^{j,k}h_{j,k} = P_{j,j'}e_{j'}\delta_{j',k'}O_{k,k}h_{j,k}$.

$$\begin{aligned} &= -\operatorname{tr}(aD^2u) \\ &\geq 0 \end{aligned}$$

This is a contradiction.

Step 2: Upgrade to all subsolutions u. Introduce the approximation

$$u_\varepsilon = u + \varepsilon v,$$

where v is a strict subsolution: $Pv < 0$ with $v \in C^2(U) \cap C(\overline{U})$. Then $u_\varepsilon \to u$ uniformly on $\overline{U}$, and

$$Pu_\varepsilon = Pu + \varepsilon Pv \leq \varepsilon Pv < 0.$$

How do we construct a strict subsolution v? We want something that is convex. A good candidate is $v = e^{x^1}$ because

$$-a^{j,k}\partial_j\partial_k(e^{x^1}) = -a^{1,1}e^{x^1} < 0.$$

We want to introduce a function which has a second order derivative much smaller than a first order derivative. So instead consider $e^{\mu x^1}$, where μ is large. Then

$$-a^{j,k}\partial_j\partial_k(e^{\mu x^1}) = -a^{1,1}e^{\mu x^1} \leq -\lambda\mu^2 e^{\mu x^1},$$

$$|b^j\partial_j e^{\mu x^1}| = |-b^j\mu e^{\mu x^1}| \leq \sup|b| \cdot \mu e^{\mu x^1}.$$

So if μ is large, $Pv < 0$. $\square$

Definition 1.2. We say that $u \in C^2(U)$ is a **(classical) supersolution** if $Pu \geq 0$.

1.2 The weak minimum principle, extension of the weak maximum principle, and the comparison principle

Theorem 1.3 (Weak minimum principle)**.** *Have the same hypotheses except assume that $Pu \geq 0$ and $c = 0$. Then*

$$\min_{\overline{U}} u = \min_{\partial U} u.$$

Remark 1.2. u is a solution if and only if it is a subsolution and a super solution. So under the same hypotheses with $Pu = 0$, we get

$$\max_{\overline{U}} |u| = \max_{\partial U} |u|.$$

Corollary 1.1 (Weak maximum principle, $c \geq 0$)**.** *Suppose U is a bounded, open connected subset of $\mathbb{R}^d$ and $u \in C^2(U) \cap C(\overline{U})$. For $Pu \leq 0$.*

$$Pu \leq 0 \implies \max_{\overline{U}} \leq \max_{\partial U} u^+,$$

$$Pu \geq 0 \implies \min_{\overline{U}} \leq \min_{\partial U} u^-,$$

where

$$u^+ = \begin{cases} u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0, \end{cases} \qquad u^+ = \begin{cases} 0 & \text{if } u \geq 0 \\ -u & \text{if } u < 0. \end{cases}$$

Proof. Here is the max part: Let $V = \{x \in U : u(x) > 0\}$, and let $Qu = Pu - cu$. Q satisfies the hypotheses and has no zero order term: $u \leq -cu \leq 0$ in V. The weak maximum principle for Q on V gives $\max_{\overline{V}} u \leq \max_{\partial V} u$. Note that the maximum of u on ∂V is the maximum of u on ∂U. So we get the claim. □

Theorem 1.4 (Comparison principle)**.** *Let U be an open, bounded, connected subset of $\mathbb{R}^d$. Let P be elliptic with $c \geq 0$. Suppose $u, v \in C^2(U) \cap C(\overline{U})$ with $Pu \leq 0$ in U and $Pv \geq 0$ in U. If $U \leq v$ on ∂U, then $u \leq v$ on U.*

Proof. This is an application of the previous corollary to $u - v$, which is a subsolution. □

1.3 The strong maximum principle

Theorem 1.5 (Strong maximum principle)**.** *Let U be an open, bounded, connected subset of $\mathbb{R}^d$, and let $c = 0$. Let $u \in C^2(U) \cap C(\overline{U})$ be such that $Pu \leq 0$. If u has a maximum at $x_0 \in U$ ($u(x) - \max_{\overline{U}} u$), then u is constant on U.*

Think of the picture of convex functions. The only way to have a maximum in the interior is if the whole function is constant (the graph is a horizontal straight line).

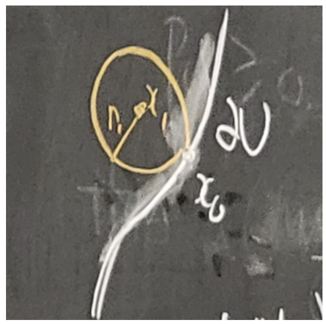
Theorem 1.6 (Hopf's lemma)**.** *Let U be an open, bounded, connected subset of $\mathbb{R}^d$. Suppose that $x_0 \in \partial U$ is such that*

(i) there exists some $x_1 \in U$ and $r_1 > 0$ such that $B_{r_1}(x_1) \subseteq U$ and $\overline{B_{r_1}(x_1)} \cap \partial U = \{x_0\}$,

(ii) $u(x_0) \geq u(x)$ in $\overline{B_{r_1}(x_1)}$,

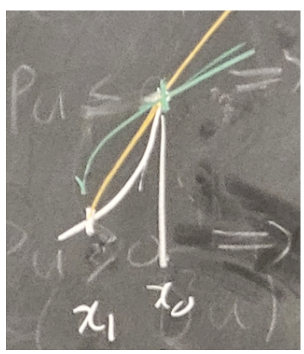
(iii) $u(x_0) > u(x)$ in $B_{r_1}(x_1)$.

Then the normal derivative $\frac{\partial}{\partial \nu}|_{x=x_0} > 0$.



Remark 1.3. We should already be able to tell that $\frac{\partial}{\partial \nu}|_{x=x_0} \geq 0$. The real content of the theorem is the strict positivity.

In the picture of convex functions, take an interior point x_1 and look at the chord connecting x_1 and the boundary point.



The idea is that this chord must have positive slope, so the actual slope of the original function at that point should be greater than the slope of the chord.

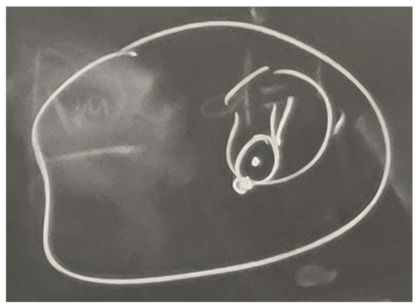
Proof. Without loss of generality, take $x_1 = 0$. Consider $v = e^{-\mu r_1^2} - e^{\mu |x|^2}$ so that $v(x) = 0$ on $\{|x| = r_1\}$. Then $Pv \geq 0$ on $B_{r_1} \setminus B_{r_1/2}$ for large μ (this is the same type of computation as before). Try to compare u to $w = v + u(x_0)$, where

$$Pw = Pv + Pu(x_0) = Pv \geq 0.$$

Let $V = B_{r_1} \setminus B_{r_1/2}$, so $\partial V = \partial B_{r_1} \cup \partial B_{r_1/2}$. On the outer boundary ∂B_{r_1}, $w = u(x_0) \geq u$. On the inner boundary $\partial B_{r_1/2}$, $w = \varepsilon v + u(x_0)$. So for small enough ε, on the inner boundary, $u(x_0) > u(x) + \varepsilon(-v)$. By the comparison principle, $w \geq u$ on $V = B_{r_1} \setminus B_{r_1/2}$. Thus,

$$\left.\frac{\partial u}{\partial \nu}\right|_{x=x_0} \geq \left.\frac{\partial v}{\partial \nu}\right|_{x=x_0} > 0. \qquad \square$$

Proof. Let $V = \{x \in U : u(x) \leq M\}$, where $M = \sup_{\overline{U}} u$. Then for $x_0 \in U$, if $u(x_0) = M$, then $V \subsetneq U$. Assume for contradiction that $V \neq \varnothing$. Find a point x_1 closer to ∂V than ∂U and consider the biggest r_1 such that $B_{r_1}(x_1) \subseteq V$. Let $x_0 \in B_{r_1}(x_1) \cap \partial V$. Let $x_0' \in B_{r_1}(x_1) \cap \partial V$.



We may arrange, by taking x_1 close enough to ∂V, so that Hopf's lemma is applicable. This tells us that $\frac{\partial}{\partial \nu} u|_{x=x_0'} \neq 0$. But this contradicts the facr that $u(x_0') = M$ implies $Du|_{x=x_0'} = 0$ $\square$